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On the dual variable of the logarithmic strain tensor, the dual variable of the Cauchy stress tensor, and related issues

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Abstract

The paper discusses relations related to the dual variables of the Cauchy stress tensor and the logarithmic strain tensor. We give a new proof that the logarithmic strain is dual to the Cauchy stress tensor in the isotropic hyperelastic case. Corresponding relations are given in the case of multiplicative finite strain elasto-viscoplasticity for the material Eshelby-like stress tensor and the rotated material stress tensor as well. The general case of anisotropic response is discussed. The dual variable of the logarithmic strain tensor is derived and some new expressions are given. Specifically it is proven that the skew-symmetric part of the Eshelby tensor is determined by the commutator of the logarithmic strain tensor and its conjugate variable. The question of the existence of dual variables for the Cauchy stress tensor and its material counterparts is rigorously answered, where it is shown that, in the general case, a dual variable does not exist. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The logarithmic strain tensor was introduced by Hencky (see e.g. Hencky (1951)) and formulated first for the one-dimensional case. In that case, it is directly related to the Cauchy stress tensor, which makes its use very appealing. In the general multi-dimensional case, specific relations where reported by Hill (1968) and a general expression for its conjugate stress in terms of the Cauchy stress tensor and the eigenvalues of the stretch tensor was derived by Hoger (1987) using spectral decomposition technique.

Strongly related to the logarithmic strains are issues concerning the conjugate strain of the Cauchy stress tensor itself. The latter generates and is strongly connected to the rotated material stress tensor as well as to an Eshelby tensor-like quantity; the latter being fundamental in the field of configurational forces. Both material stress tensors can be viewed as generated by mixed-variant transformations of the Kirchhoff stress tensor.

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The paper discusses the existence of a conjugate strain for the aforementioned stress quantities. A new and alternative proof is given for Hill's result that, in the isotropic hyperelastic case, the Cauchy stress tensor is conjugate to the logarithm of the left stretch tensor.

In the isotropic case, the treatment is extended to encompass multiplicative finite strain viscoplasticity. Here the corresponding conjugate strains of the aforementioned stress tensors are derived. Remarkably, unlike the purely hyperelastic case, the conjugate strains do not coincide.

The general anisotropic case is discussed. First, new expressions for the conjugate stress tensor of the logarithmic strain is given. Moreover, a closed formula for the skew-symmetric part of the Eshelby-like material stress in terms of the logarithmic strain and its conjugate stress is derived.

The developments make no use of spectral decompositions whatsoever. The treatment rests on exploiting the features of the series expansion of the exponential map. As a by-product, different new relations related to it are derived.

Moreover, a rigor proof that none of the above-mentioned stress tensors, the Kirchhoff (or Cauchy), the rotated stress tensor, and the Eshelby-like one does have conjugate strains in the anisotropic case, is given.

The paper is organized as follows. In Section 2 basic relations of continuum mechanics are recalled. In Section 3.1 the isotropic case is treated. The dual variable of the Cauchy stress tensor is derived. First for the case of hyperelasticity and then for that of finite strain multiplicative inelasticity. The results are extended to encompass the material Eshelby-like tensor as well as the rotated material stress tensor. Section 3.2 deals with the anisotropic case. The dual variable of the logarithmic strain tensor is provided. Moreover, a general proof for the non-existence of a conjugate variable for the Cauchy stress tensor (or its related material tensors) is given. A discussion is provided in Section 4. The paper closes is Section 5.

Throughout the paper the words dual and conjugate are used synonymously. We stress that the use of the terminology of duality is not uniform in the literature. We stick here to the definition of Hill (1978) and note that different definitions exist, see e.g. Haupt and Tsakmakis (1989) for an alternative definition. We refer too to Xiao et al. (1997) for the discussion of further issues of the logarithmic strains in conjunction with a newly defined so-called logarithmic rate.

2. Preliminaries

Let $\mathscr{B} \subset \mathscr{R}^3$ define a body. A motion of the body $\mathscr{B}$ is represented by a one parameter mapping $\boldsymbol{\varphi}_t : \mathscr{B} \rightarrow \mathscr{B}_t$, where $t \in \mathscr{R}$ is the time and $\mathscr{B}_t$ is the current configuration at time t. Associated with each material point of the body $\mathscr{B}$ are the position vectors $\mathbf{X} \in \mathscr{B}$ at the reference configuration and $\mathbf{x} \in \mathscr{B}_t$ at the current configuration. One has $\boldsymbol{\varphi}_t(\mathbf{X}) = \mathbf{x} \in \mathscr{B}_t$. In the following explicit reference to t will be omitted. The tangent map related to $\boldsymbol{\varphi}$ is the deformation gradient $\mathbf{F}$ which maps the tangent space $T_X\mathscr{B}$ at the reference configuration to the tangent space $T_x\mathscr{B}_t$ at the actual configuration: $\mathbf{F} := T_X\mathscr{B} \rightarrow T_x\mathscr{B}_t$. The deformation gradient is a two-point tensor.

With the help of $\mathbf{F}$ one defines the right Cauchy–Green strain tensor

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}. \tag{1}$$

In addition, the deformation gradient posses the polar decompositions

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \quad \mathbf{F} = \mathbf{V}\mathbf{R} \quad \text{and} \quad \mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}} \tag{2}$$

with $\mathbf{U}$ and $\mathbf{V}$ being symmetric and $\mathbf{R} \in \mathrm{SO}(3, \mathscr{R})$, the space of orthogonal tensors with positive determinants. By the symmetry $\mathbf{U}$ and $\mathbf{C}$ and the fact $\det \mathbf{U} > 0$ the logarithmic strain tensor $\boldsymbol{\alpha}$ can be introduced according to

$$\boldsymbol{\alpha} = \ln \mathbf{U} = \tfrac{1}{2} \ln \mathbf{C}. \tag{3}$$

Alternatively we have

$$\mathbf{U} = \exp \boldsymbol{\alpha} = \mathbf{1} + \boldsymbol{\alpha} + \frac{\boldsymbol{\alpha}^2}{2!} + \frac{\boldsymbol{\alpha}^3}{3!} + \frac{\boldsymbol{\alpha}^4}{4!} + \cdots \tag{4}$$

and, with $\boldsymbol{\beta} = 2\boldsymbol{\alpha}$,

$$\mathbf{C} = \exp \boldsymbol{\beta} = \mathbf{1} + \boldsymbol{\beta} + \frac{\boldsymbol{\beta}^2}{2!} + \frac{\boldsymbol{\beta}^3}{3!} + \frac{\boldsymbol{\beta}^4}{4!} + \cdots \tag{5}$$

Viewing the deformation gradient as an element of the Lie group $GL^+(3,\mathscr{R})$ (the general linear transformation group with positive determinants) one can define a left and a right rate according to

$$\dot{\mathbf{F}} = \mathbf{lF} = \mathbf{FL} \tag{6}$$

with

$$\mathbf{l} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad \mathbf{L} = \mathbf{F}^{-1}\dot{\mathbf{F}}, \quad \mathbf{l} = \mathbf{FLF}^{-1}, \tag{7}$$

where a dot denotes time derivative. The symmetric parts of $\mathbf{l}$ and $\mathbf{L}$ are given by

$$\mathbf{d} = \tfrac{1}{2}(\mathbf{l} + \mathbf{l}^{\mathrm{T}}), \qquad \mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}). \tag{8}$$

Let $\boldsymbol{\sigma}$ be the Cauchy stress tensor, $\boldsymbol{\tau}$ the Kirchhoff stress tensor, $\mathbf{S}$ the second Piola–Kirchhoff stress tensor, and ρ, ρ_{ref} the densities at the actual and reference configurations, respectively. The relations hold

$$\boldsymbol{\tau} = \frac{\rho_{\mathrm{ref}}}{\rho}\boldsymbol{\sigma}, \qquad \boldsymbol{\tau} = \mathbf{FSF}^{\mathrm{T}}. \tag{9}$$

Now, we express the rate of the specific internal work $\mathscr{W}$ in terms of the spatial and material velocity gradients defined in Eq. (6). One has

$$\mathscr{W} = \boldsymbol{\tau} : \mathbf{l} = \Xi : \mathbf{L}, \tag{10}$$

where (:) defines the scalar product of two tensors ($\boldsymbol{\sigma}{:}\mathbf{d} = \mathrm{tr}(\boldsymbol{\sigma}\mathbf{d}^{\mathrm{T}})$, with tr denoting the trace operation). With Eq. (7) (third form) the material stress tensor Ξ is defined as

$$\Xi = \mathbf{F}^{\mathrm{T}}\boldsymbol{\tau}\mathbf{F}^{-\mathrm{T}}. \tag{11}$$

Up to a sign and a spherical part it coincides with the Eshelby stress tensor (see e.g. Maugin (1995)). In the following we refer to it as an Eshelby-like stress tensor. One has also the alternative relation

$$\Xi = \mathbf{CS}. \tag{12}$$

In addition the rotated stress tensor

$$\boldsymbol{\Sigma} = \mathbf{R}^{\mathrm{T}}\boldsymbol{\tau}\mathbf{R} \tag{13}$$

is introduced. The relations hold

$$\Xi = \mathbf{U}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{U}^{-\mathrm{T}} \tag{14}$$

and

$$\Sigma = \mathbf{U}^{\mathrm{T}}\mathbf{SU}. \tag{15}$$

The angular momentum equation is equivalent to a symmetry condition imposed on $\boldsymbol{\sigma}$, respectively on $\boldsymbol{\tau}$. The same holds true for the symmetry of $\mathbf{S}$ and Σ. The same restriction can be expressed in terms of the, in general, non-symmetric tensor Ξ which leads to the relation

$$\Xi^{\mathrm{T}} = \mathbf{C}^{-1}\Xi\mathbf{C}. \tag{16}$$

This restriction reduces Eq. (10) (first form) to

$$\mathscr{W} = \boldsymbol{\tau} : \mathbf{d}. \tag{17}$$

The corresponding expression in terms of Ξ reduces with Eq. (16) to

$$\mathscr{W} = \tfrac{1}{2}\Xi : (\mathbf{L} + \mathbf{C}^{-1}\mathbf{L}^{\mathrm{T}}\mathbf{C}). \tag{18}$$

Note that while the symmetric part of $\mathbf{l}$ enters the formula, the same quantity related to $\mathbf{L}$ does not play the same role. In addition, in terms of the rotated stress tensor one has

$$\mathscr{W} = \frac{1}{2}\Sigma : (\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1}), \tag{19}$$

where the symmetry of Σ has been employed.

3. Dual variables

In terms of the second Piola–Kirchhoff stress tensor, $\mathscr{W}$ can be formulated as

$$\mathscr{W} = \frac{1}{2}\mathbf{S} : \dot{\mathbf{C}}. \tag{20}$$

A fundamental difference between the last relation and those written down in terms of the three stress tensors $\boldsymbol{\tau}, \Sigma$, and Ξ is the fact that, in Eq. (20), the rate of a strain tensor is incorporated where, in contrast, $\mathbf{l}, \mathbf{L}$ do not constitute such a quantity. We deduce from Eq. (20) that $\mathbf{S}$ and $\mathbf{C}$ are dual variables. Equivalently, given an internal potential $\psi_{\mathrm{int}}(\mathbf{C}, \ldots)$ depending on $\mathbf{C}$ and, possibly, on further yet unspecified internal variables symbolized here with dots, one has the well known relation

$$\mathbf{S} = 2\rho_{\mathrm{ref}}\frac{\partial\psi_{\mathrm{int}}(\mathbf{C}, \ldots)}{\partial\mathbf{C}}. \tag{21}$$

Now since $\mathbf{l}$ or $\mathbf{d}$ themselves are not the rates of a strain tensor one may ask whether there exist strain measures, yet unknown functions of $\mathbf{U}$ or $\mathbf{C}$, which can be regarded as dual variables for $\boldsymbol{\tau}$, Σ, or Ξ, respectively. We answer this question in two steps. We consider first the isotropic case and then extend the treatment to the non-isotropic case.

3.1. The isotropic case

Two cases, (1) isotropic hyperelasticity and (2) isotropic inelasticity are considered. In the first case one is assuming the internal potential to depend on $\mathbf{C}$ itself, while in the second case one has to introduce an adequate elastic strain measure. The internal potential is then considered to be an isotropic function of the latter.

3.1.1. Hyperelasticity

Proposition 1. *The logarithmic strain tensor* $\boldsymbol{\alpha}$ *is the dual variable of* $\boldsymbol{\Sigma}$ *and of* $\boldsymbol{\Xi}$. *The dual variable of the Kirchhoff stress tensor is then* $\mathbf{R}\boldsymbol{\alpha}\mathbf{R}^{\mathrm{T}} = \ln\mathbf{V}$.

Proof. This fact was already known to Hill (1968). A general proof using spectral decomposition technique was presented by Hoger (1987). We give here an alternative and very direct proof using Eq. (5)

We have to prove that we can write for Eq. (10): $\mathscr{W} = \boldsymbol{\Sigma}{:}\dot{\boldsymbol{\alpha}} = \boldsymbol{\Xi}{:}\dot{\boldsymbol{\alpha}}$. First we have

$$\dot{\mathbf{C}} = \dot{\boldsymbol{\beta}} + \frac{1}{2!}(\boldsymbol{\beta}\dot{\boldsymbol{\beta}} + \dot{\boldsymbol{\beta}}\boldsymbol{\beta}) + \frac{1}{3!}(\dot{\boldsymbol{\beta}}\boldsymbol{\beta}^2 + \boldsymbol{\beta}^2\dot{\boldsymbol{\beta}} + \boldsymbol{\beta}\dot{\boldsymbol{\beta}}\boldsymbol{\beta}) + \cdots \tag{22}$$

For the specific rate of the internal work one has

$$\frac{1}{2}\mathbf{S} : \dot{\mathbf{C}} = \frac{1}{2}\mathbf{S} : \left(\dot{\boldsymbol{\beta}} + \frac{1}{2!}(\boldsymbol{\beta}\dot{\boldsymbol{\beta}} + \dot{\boldsymbol{\beta}}\boldsymbol{\beta}) + \frac{1}{3!}(\dot{\boldsymbol{\beta}}\boldsymbol{\beta}^2 + \boldsymbol{\beta}^2\dot{\boldsymbol{\beta}} + \boldsymbol{\beta}\dot{\boldsymbol{\beta}}\boldsymbol{\beta}) + \cdots\right), \tag{23}$$

$$= \frac{1}{2}\left(\mathbf{S} + \frac{1}{2!}(\mathbf{S}\boldsymbol{\beta} + \boldsymbol{\beta}\mathbf{S}) + \frac{1}{3!}(\mathbf{S}\boldsymbol{\beta}^2 + \boldsymbol{\beta}\mathbf{S}\boldsymbol{\beta} + \boldsymbol{\beta}^2\mathbf{S}) + \cdots\right) : \dot{\boldsymbol{\beta}}. \tag{24}$$

Assuming now that $\mathbf{S}$ is an isotropic function of $\mathbf{C}$, and hence of $\boldsymbol{\beta}$ as well, which essentially means that $\mathbf{S}$ commutes with $\boldsymbol{\beta}$, the last equation renders

$$\mathscr{W} = \frac{1}{2}\left(\mathbf{1} + \boldsymbol{\beta} + \frac{\boldsymbol{\beta}^2}{2!} + \frac{\boldsymbol{\beta}^3}{3!} + \cdots\right)\mathbf{S} : \dot{\boldsymbol{\beta}} \tag{25}$$

$$= \frac{1}{2}\mathbf{CS} : \dot{\boldsymbol{\beta}} \tag{26}$$

$$= \Xi : \dot{\boldsymbol{\alpha}} \tag{27}$$

which proves that Ξ and $\boldsymbol{\alpha}$ are dual variables. Use has been made of Eqs. (5) and (12).

On the other hand, Ξ is an isotropic function of $\mathbf{U}$ as well. From Eq. (14) we immediately infer that $\Xi = \Sigma$ and hence $\boldsymbol{\alpha}$ is the dual variable of Σ as well which, with Eq. (13), conclude the proof. □

For the extension of the above considerations to inelastic processes, it proves helpful to consider a proof using the existence of an internal potential which we consider now. Accordingly, the internal potential ψ is an isotropic function of $\mathbf{C}$. Essentially one has to prove that the relation holds

$$\frac{\partial\psi}{\partial\boldsymbol{\beta}} = \frac{\partial\psi}{\partial\mathbf{C}} : \frac{\partial\mathbf{C}}{\partial\boldsymbol{\beta}} = \mathbf{C}\frac{\partial\psi}{\partial\mathbf{C}}. \tag{28}$$

Since $\mathbf{C} = \exp\boldsymbol{\beta}$ from the Taylor series representation of the exponential map the left-hand side of Eq. (28) takes

$$\begin{aligned}\frac{\partial\psi}{\partial\mathbf{C}} : \frac{\partial\mathbf{C}}{\partial\boldsymbol{\beta}} &= \frac{\partial\psi}{\partial\mathbf{C}} + \frac{1}{2}\frac{\partial\psi}{\partial\mathbf{C}}\boldsymbol{\beta}^{\mathrm{T}} + \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}}\frac{\partial\psi}{\partial\mathbf{C}} + \frac{1}{3!}\frac{\partial\psi}{\partial\mathbf{C}}\boldsymbol{\beta}^{2\mathrm{T}} + \frac{1}{3!}\boldsymbol{\beta}^{2\mathrm{T}}\frac{\partial\psi}{\partial\mathbf{C}} + \frac{1}{3!}\boldsymbol{\beta}^{\mathrm{T}}\frac{\partial\mathbf{C}}{\partial\boldsymbol{\beta}}\boldsymbol{\beta}^{\mathrm{T}} + \cdots \\ &= \frac{\partial\psi}{\partial\mathbf{C}} + \boldsymbol{\beta}^{\mathrm{T}}\frac{\partial\psi}{\partial\mathbf{C}} + \frac{1}{2!}\boldsymbol{\beta}^{2\mathrm{T}}\frac{\partial\psi}{\partial\mathbf{C}} + \cdots = \mathbf{C}\frac{\partial\psi}{\partial\mathbf{C}}.\end{aligned} \tag{29}$$

The last relation taken together with Eqs. (12) and (21) proves the following statement

$$\Xi = \rho_{\mathrm{ref}}\frac{\partial\psi}{\partial\boldsymbol{\alpha}}, \tag{30}$$

which indicates that Ξ and $\boldsymbol{\alpha}$ are dual variables.

3.1.2. Multiplicative viscoplasticity

In the case of isotropic multiplicative viscoplasticity things take a slightly different shape. Most important is the fact that Ξ and Σ do not coincide. For the seek of completeness we review some of the results obtained in (Sansour and Kollmann, 1997) and extend the treatment to include the rotated and the Kirchhoff stress tensor.

We start with the classical decomposition

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p \tag{31}$$

define the right Cauchy–Green-type tensors

$$\mathbf{C}_e := \mathbf{F}_e^T\mathbf{F}_e, \tag{32}$$

$$\mathbf{C}_p := \mathbf{F}_p^T\mathbf{F}_p, \tag{33}$$

and assume the existence of a free energy function of the form $\psi = \psi(\mathbf{C}_e, \mathbf{Z})$, where $\mathbf{Z}$ defines a vector of internal variables (independent of the elastic strain). Straightforward calculations lead to

$$\mathbf{S} = 2\rho_{ref}\mathbf{F}_p^{-1}\frac{\partial\psi(\mathbf{C}_e, \mathbf{Z})}{\partial\mathbf{C}_e}\mathbf{F}_p^{-T} \tag{34}$$

and

$$\Xi = 2\rho_{ref}\mathbf{C}\mathbf{F}_p^{-1}\frac{\partial\psi(\mathbf{C}_e, \mathbf{Z})}{\partial\mathbf{C}_e}\mathbf{F}_p^{-T}. \tag{35}$$

The latter equation can be recast in the form

$$\Xi = 2\rho_{ref}\mathbf{F}_p^T\mathbf{C}_e\frac{\partial\psi(\mathbf{C}_e, \mathbf{Z})}{\partial\mathbf{C}_e}\mathbf{F}_p^{-T}. \tag{36}$$

Further we assume that the elastic potential can be decomposed additively into one part depending only on the elastic right Cauchy–Green deformation tensor $\mathbf{C}_e$ and the other one depending only on the internal variable $\mathbf{Z}$

$$\psi = \psi_e(\mathbf{C}_e) + \psi_Z(\mathbf{Z}). \tag{37}$$

For Eq. (36) we have accordingly

$$\Xi = 2\rho_{ref}\mathbf{F}_p^T\mathbf{C}_e\frac{\partial\psi_e(\mathbf{C}_e)}{\partial\mathbf{C}_e}\mathbf{F}_p^{-T}. \tag{38}$$

We are ready now for the following proposition.

Proposition 2. *The tensor*

$$\bar{\boldsymbol{\alpha}}_e = \tfrac{1}{2}\ln(\mathbf{C}_p^{-1}\mathbf{C}) \tag{39}$$

constitutes the dual variable of Ξ.

Proof. First, the logarithmic of $\mathbf{C}_e$ is defined

$$\boldsymbol{\alpha}_e := \tfrac{1}{2}\boldsymbol{\beta}_e = \tfrac{1}{2}\ln\,\mathbf{C}_e, \quad \mathbf{C}_e = \exp\boldsymbol{\beta}_e. \tag{40}$$

In accordance with Eq. (28), one has

$$\mathbf{C}_e\frac{\partial\psi_e(\mathbf{C}_e)}{\partial\mathbf{C}_e} = \frac{\partial\psi_e(\boldsymbol{\beta}_e)}{\partial\boldsymbol{\beta}_e}, \tag{41}$$

where $\psi_e(\boldsymbol{\beta}_e)$ is the potential expressed in the logarithmic strain measure $\boldsymbol{\beta}_e$. Eq. (38) results then in

$$\Xi = 2\rho_{ref}\mathbf{F}_p^T\frac{\partial\psi_e(\boldsymbol{\beta}_e)}{\partial\boldsymbol{\beta}_e}\mathbf{F}_p^{-T}. \tag{42}$$

The Cayley–Hamilton theorem for isotropic tensor functions yields

$$\Xi = 2\rho_{\mathrm{ref}}\mathbf{F}_{\mathrm{p}}^{\mathrm{T}}\Big(\gamma_0\mathbf{1} + \gamma_1 \exp\boldsymbol{\beta}_{\mathrm{e}} + \gamma_2(\exp\boldsymbol{\beta}_{\mathrm{e}})^2\Big)\mathbf{F}_{\mathrm{p}}^{-\mathrm{T}}, \tag{43}$$

where γ_{i}, $\mathrm{i} = 0, 1, 2$ are functions of the invariants of $\exp\boldsymbol{\beta}_{\mathrm{e}}$. Eq. (43) motivates the introduction of a modified logarithmic strain measure

$$\overline{\boldsymbol{\beta}}_{\mathrm{e}} := \mathbf{F}_{\mathrm{p}}^{-1}\boldsymbol{\beta}_{\mathrm{e}}\mathbf{F}_{\mathrm{p}}, \quad \overline{\boldsymbol{\alpha}}_{\mathrm{e}} := \tfrac{1}{2}\overline{\boldsymbol{\beta}}_{\mathrm{e}}. \tag{44}$$

Considering the series expansion of the exponential map it can be shown that

$$\mathbf{F}_{\mathrm{p}}^{-1}(\exp\boldsymbol{\beta}_{\mathrm{e}})\mathbf{F}_{\mathrm{p}} = \exp\overline{\boldsymbol{\beta}}_{\mathrm{e}} \tag{45}$$

and therefore Eq. (42) takes

$$\Xi = 2\rho_{\mathrm{ref}}\frac{\partial\psi(\overline{\boldsymbol{\beta}}_{\mathrm{e}})}{\partial(\overline{\boldsymbol{\beta}}_{\mathrm{e}})} = \rho_{\mathrm{ref}}\frac{\partial\psi(\overline{\boldsymbol{\alpha}}_{\mathrm{e}})}{\partial\overline{\boldsymbol{\alpha}}_{\mathrm{e}}}. \tag{46}$$

Accordingly, Ξ and $\overline{\boldsymbol{\alpha}}_{\mathrm{e}}$ are dual variables.

The relations (1), (33), (40) and (48) lead to a direct definition of $\overline{\boldsymbol{\alpha}}_{\mathrm{e}}$ as

$$\overline{\boldsymbol{\alpha}}_{\mathrm{e}} = \tfrac{1}{2}\ln(\mathbf{C}_{\mathrm{p}}^{-1}\mathbf{C}) \tag{47}$$

which completes the proof. □

Proposition 3. *Let*

$$\boldsymbol{\alpha}_{\mathrm{e}}^{*} = \tfrac{1}{2}\ln(\mathbf{U}\mathbf{C}_{\mathrm{p}}^{-1}\mathbf{U}^{\mathrm{T}}). \tag{48}$$

Then $\boldsymbol{\alpha}_{\mathrm{e}}^{*}$ and the rotated stress tensor $\boldsymbol{\Sigma}$ are dual variables.

Proof. First we observe that Eq. (48) can be reformulated as

$$\boldsymbol{\alpha}_{\mathrm{e}}^{*} = \mathbf{U}\overline{\boldsymbol{\alpha}}_{\mathrm{e}}\mathbf{U}^{-1}. \tag{49}$$

Understanding ψ as a function of $\boldsymbol{\alpha}_{\mathrm{e}}^{*}$, Eq. (46), together with the last equation, yields

$$\Xi = \rho_{\mathrm{ref}}\frac{\partial\psi(\boldsymbol{\alpha}_{\mathrm{e}}^{*})}{\partial\boldsymbol{\alpha}_{\mathrm{e}}^{*}}\frac{\partial\boldsymbol{\alpha}_{\mathrm{e}}^{*}}{\partial\overline{\boldsymbol{\alpha}}_{\mathrm{e}}} = \rho_{\mathrm{ref}}\mathbf{U}^{\mathrm{T}}\frac{\partial\psi(\boldsymbol{\alpha}_{\mathrm{e}}^{*})}{\partial\boldsymbol{\alpha}_{\mathrm{e}}^{*}}\mathbf{U}^{-\mathrm{T}}. \tag{50}$$

Accordingly we have with reference to Eq. (14) immediately

$$\Sigma = \rho_{\mathrm{ref}}\frac{\partial\psi(\boldsymbol{\alpha}_{\mathrm{e}}^{*})}{\partial\boldsymbol{\alpha}_{\mathrm{e}}^{*}} \tag{51}$$

concluding the proof. □

Proposition 4. *The dual variable of the Kirchhoff stress tensor is*

$$\tilde{\boldsymbol{\alpha}}_{\mathrm{e}} = \tfrac{1}{2}\ln(\mathbf{F}\mathbf{C}_{\mathrm{p}}^{-1}\mathbf{F}^{\mathrm{T}}). \tag{52}$$

Proof. Here also we notice that Eq. (52) can be rewritten as

$$\tilde{\boldsymbol{\alpha}}_{\mathrm{e}} = \mathbf{R}\boldsymbol{\alpha}_{\mathrm{e}}^{*}\mathbf{R}^{\mathrm{T}}, \tag{53}$$

which again, with Eq. (13), immediately leads to

$$\boldsymbol{\tau} = \rho_{\mathrm{ref}} \frac{\partial \psi(\tilde{\boldsymbol{\alpha}}_{\mathrm{e}})}{\partial \tilde{\boldsymbol{\alpha}}_{\mathrm{e}}}. \tag{54}$$

Accordingly, $\boldsymbol{\tau}$ is dual to $\tilde{\boldsymbol{\alpha}}_{\mathrm{e}}$. □

It is worthy noting that in contrast to the hyperelastic case, the dual variables of Ξ and Σ do not coincide.

3.2. The anisotropic case

3.2.1. The dual variable of the logarithmic strain

We start again with Eq. (5) and derive first the dual variable of the logarithmic strain tensor. The proof is general as we are not going to make use of any assumptions regarding the isotropy of the material law. One has

$$\mathscr{W} = \frac{1}{2}\mathbf{S} : \dot{\mathbf{C}} \tag{55}$$

$$= \frac{1}{2}\mathbf{S} : \left(\dot{\boldsymbol{\beta}} + \frac{1}{2!}(\boldsymbol{\beta}\dot{\boldsymbol{\beta}} + \dot{\boldsymbol{\beta}}\boldsymbol{\beta}) + \frac{1}{3!}(\dot{\boldsymbol{\beta}}\boldsymbol{\beta}^2 + \boldsymbol{\beta}^2\dot{\boldsymbol{\beta}} + \boldsymbol{\beta}\dot{\boldsymbol{\beta}}\boldsymbol{\beta}) + \cdots\right) \tag{56}$$

$$= \frac{1}{2}\left(\mathbf{S} + \frac{1}{2!}(\mathbf{S}\boldsymbol{\beta} + \boldsymbol{\beta}\mathbf{S}) + \frac{1}{3!}(\mathbf{S}\boldsymbol{\beta}^2 + \boldsymbol{\beta}\mathbf{S}\boldsymbol{\beta} + \boldsymbol{\beta}^2\mathbf{S}) + \cdots\right) : \dot{\boldsymbol{\beta}} \tag{57}$$

$$= \Sigma_{\ln} : \dot{\boldsymbol{\alpha}} \tag{58}$$

Hence the dual variable of the logarithmic strain reads

$$\Sigma_{\ln} = \mathbf{S} + \frac{1}{2!}(\mathbf{S}\boldsymbol{\beta} + \boldsymbol{\beta}\mathbf{S}) + \frac{1}{3!}(\mathbf{S}\boldsymbol{\beta}^2 + \boldsymbol{\beta}\mathbf{S}\boldsymbol{\beta} + \boldsymbol{\beta}^2\mathbf{S}) + \cdots \tag{59}$$

It is obvious that, by the symmetry of $\mathbf{S}$ and $\boldsymbol{\beta}$, $\boldsymbol{\Sigma}_{\ln}$ itself is symmetric as well.

It is fruitful to express $\boldsymbol{\Sigma}_{\ln}$ in terms of Ξ and $\boldsymbol{\beta}$. For this reason we introduce the Lie bracket. For any tensors $\mathbf{A}, \mathbf{B}$ the Lie bracket (the commutator) is defined as

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}. \tag{60}$$

Evidently, the Lie bracket is a measure of the non-commutativity of two tensor. By its very definition the following features of the Lie bracket hold: ($\mathbf{A}_{\mathrm{sym}}$ and $\mathbf{A}_{\mathrm{skew}}$ are any symmetric or skew-symmetric tensors, respectively the corresponding symmetric or skew-symmetric parts of a tensor)

$$[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}], \tag{61}$$

$$[\mathbf{A}_{\mathrm{sym}}, \mathbf{B}_{\mathrm{sym}}]^{\mathrm{T}} = -[\mathbf{A}_{\mathrm{sym}}, \mathbf{B}_{\mathrm{sym}}], \tag{62}$$

$$[\mathbf{A}_{\mathrm{sym}}, \mathbf{B}_{\mathrm{skew}}]^{\mathrm{T}} = [\mathbf{A}_{\mathrm{sym}}, \mathbf{B}_{\mathrm{skew}}], \tag{63}$$

$$[\mathbf{A}_{\mathrm{skew}}, \mathbf{B}_{\mathrm{skew}}]^{\mathrm{T}} = -[\mathbf{A}_{\mathrm{skew}}, \mathbf{B}_{\mathrm{skew}}]. \tag{64}$$

Making use of $\mathbf{S} = \mathbf{C}^{-1}\boldsymbol{\Xi}$ as well as of

$$\mathbf{C}^{-1} = \mathbf{1} - \boldsymbol{\beta} + \frac{\boldsymbol{\beta}^2}{2!} - \frac{\boldsymbol{\beta}^3}{3!} + \cdots \tag{65}$$

Eq. (59) can be reformulated. After some extended but otherwise straightforward calculations one arrives at the following expression

$$\boldsymbol{\Sigma}_{\ln} = \boldsymbol{\Xi} + \frac{1}{2!}[\boldsymbol{\Xi}, \boldsymbol{\beta}] + \frac{1}{3!}[[\boldsymbol{\Xi}, \boldsymbol{\beta}], \boldsymbol{\beta}] + \frac{1}{4!}[[[\boldsymbol{\Xi}, \boldsymbol{\beta}], \boldsymbol{\beta}], \boldsymbol{\beta}] + \cdots \tag{66}$$

The expression is very remarkable. Although $\boldsymbol{\Xi}$ is not symmetric, the symmetry of $\boldsymbol{\Sigma}_{\ln}$ is hidden behind the side condition (16), the angular momentum equation. Using this side condition one can extract some interesting results. For this purpose we need some relations which we prove now.

First, for the seek of brevity we make use of the notation $[\boldsymbol{\Xi}, \boldsymbol{\beta}]_n$ for n-times successive application of the Lie bracket with respect to $\boldsymbol{\beta}$. Accordingly we write e.g. $[\boldsymbol{\Xi}, \boldsymbol{\beta}]_3$ to denote $[[[\boldsymbol{\Xi}, \boldsymbol{\beta}], \boldsymbol{\beta}], \boldsymbol{\beta}]$. The following proposition can be proven.

Proposition 5. *The relation holds*

$$[\boldsymbol{\Xi}, \boldsymbol{\beta}]_n^{\mathrm{T}} = (-1)^n \mathbf{C}^{-1}[\boldsymbol{\Xi}, \boldsymbol{\beta}]_n \mathbf{C} \tag{67}$$

Proof. For $n = 1$ one has using Eq. (16) and the symmetry of $\boldsymbol{\beta}$:

$$[\boldsymbol{\Xi}, \boldsymbol{\beta}]^{\mathrm{T}} = \boldsymbol{\beta}\boldsymbol{\Xi}^{\mathrm{T}} - \boldsymbol{\Xi}^{\mathrm{T}}\boldsymbol{\beta} = \boldsymbol{\beta}\mathbf{C}^{-1}\boldsymbol{\Xi}\mathbf{C} - \mathbf{C}^{-1}\boldsymbol{\Xi}\mathbf{C}\boldsymbol{\beta}, \tag{68}$$

$$= \mathbf{C}^{-1}\boldsymbol{\beta}\boldsymbol{\Xi}\mathbf{C} - \mathbf{C}^{-1}\boldsymbol{\Xi}\boldsymbol{\beta}\mathbf{C}, \tag{69}$$

$$= -\,\mathbf{C}^{-1}[\boldsymbol{\Xi}, \boldsymbol{\beta}]\mathbf{C}. \tag{70}$$

Use has been made of the fact that $\mathbf{C}$ is an isotropic function of $\boldsymbol{\beta}$ and hence that both quantities commute. The successive application of the above rule leads directly to the statement of the proposition. □

Proposition 6. *For any tensor* $\mathbf{A}$ *the relation holds*

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{A} + [\mathbf{A}, \boldsymbol{\beta}] + \sum_{n=2}^{\infty} \frac{1}{n!}[\mathbf{A}, \boldsymbol{\beta}]_n. \tag{71}$$

Proof. The proof follows directly by using Eqs. (5) and (65) after some extensive but otherwise elementary elaborations. □

We prove now the remarkable relation that the skew-symmetric part of the Eshelby-like tensor Ξ is given by the Lie bracket of the logarithmic strain tensor with its dual stress tensor.

Proposition 7. *The relation holds*

$$\boldsymbol{\Xi}_{\mathrm{skew}} = -\tfrac{1}{2}[\boldsymbol{\Sigma}_{\ln}, \boldsymbol{\beta}] = -[\boldsymbol{\Sigma}_{\ln}, \boldsymbol{\alpha}]. \tag{72}$$

Proof. Applying the last proposition to Ξ and with Eq. (16) we have immediately

$$\boldsymbol{\Xi}^{\mathrm{T}} = \boldsymbol{\Xi} + [\boldsymbol{\Xi}, \boldsymbol{\beta}] + \sum_{n=2}^{\infty} \frac{1}{n!}[\boldsymbol{\Xi}, \boldsymbol{\beta}]_n \tag{73}$$

or alternatively

$$\boldsymbol{\Xi}^{\mathrm{T}} - \boldsymbol{\Xi} = [\boldsymbol{\Xi}, \boldsymbol{\beta}] + \sum_{n=2}^{\infty} \frac{1}{n!}[\boldsymbol{\Xi}, \boldsymbol{\beta}]_n. \tag{74}$$

Observing Eq. (66) and taking its Lie bracket with respect to $\boldsymbol{\beta}$, we arrive directly at the remarkable relation stated in the proposition above. □

In passing we note that by the very definition of Ξ, Eq. (12), the relation is immediate

$$\mathbf{\Xi}_{\text{skew}} = -\tfrac{1}{2}[\mathbf{S}, \mathbf{C}]. \tag{75}$$

Nevertheless, that such a relation holds also true for the dual variable of the logarithmic strain is not obvious. Note also that the relation holds as well

$$\operatorname{tr} \mathbf{\Sigma}_{\text{ln}} = \operatorname{tr} \mathbf{\Sigma} = \operatorname{tr} \mathbf{\Xi}. \tag{76}$$

In the purely isotropic case the tensors $\mathbf{\Sigma}_{\text{ln}}$ and $\boldsymbol{\beta}$ commute rendering the corresponding Lie bracket zero which recovers the result that the symmetric rotated stress tensor coincides with the material Eshelby-like tensor and with $\mathbf{\Sigma}_{\text{ln}}$ itself as well.

3.2.2. The non-existence of dual variable for the Cauchy stress tensor

The specific rate of internal work can be recast in the form

$$\mathscr{W} = \tfrac{1}{2}\mathbf{S} : \dot{\mathbf{C}}, \tag{77}$$

$$= \tfrac{1}{2}\mathbf{\Xi} : \mathbf{C}^{-1}\dot{\mathbf{C}}, \tag{78}$$

$$= \frac{1}{2}\mathbf{\Xi}_{\text{sym}} : (\mathbf{C}^{-1}\dot{\mathbf{C}} + \dot{\mathbf{C}}\mathbf{C}^{-1}) + \tfrac{1}{2}\mathbf{\Xi}_{\text{skew}} : (\mathbf{C}^{-1}\dot{\mathbf{C}} - \dot{\mathbf{C}}\mathbf{C}^{-1}). \tag{79}$$

On the other hand the same expression in terms of the material rotated Kirchhoff stress tensor reads

$$\mathscr{W} = \tfrac{1}{2}\mathbf{\Sigma} : (\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1}). \tag{80}$$

We have seen that $\mathbf{\Xi}_{\text{skew}}$ is directly related to the conjugate stress of the logarithmic strain. We ask now whether there exists a strain tensor, to be considered as an isotropic function of $\mathbf{U}$, or equivalently of $\mathbf{C}$ or $\boldsymbol{\alpha}$, which can be considered as conjugate strain to the symmetric part $\mathbf{\Xi}_{\text{sym}}$. Accordingly we ask whether we can define a strain tensor, say $\mathbf{Y}$, such that $2\mathbf{\Xi}_{\text{sym}} : \dot{\mathbf{Y}} = \mathbf{\Xi}_{\text{sym}} : (\mathbf{C}^{-1}\dot{\mathbf{C}} + \dot{\mathbf{C}}\mathbf{C}^{-1})$. The same argumentation is valid for the tensor Σ since one just need to replace $\mathbf{C}$ with $\mathbf{U}$.

In the following we will prove a negative answer.

Proposition 8. *There does not exist any tensor* $\mathbf{Y}$, *function of* $\mathbf{U}$, *such that the relation holds*

$$\mathbf{\Sigma} : \dot{\mathbf{Y}} = \tfrac{1}{2}\mathbf{\Sigma} : (\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1}). \tag{81}$$

Likewise, there does not exist any tensor $\mathbf{Y}$, *function of* $\mathbf{C}$, *such that the relation holds*

$$\mathbf{\Xi}_{\text{sym}} : \dot{\mathbf{Y}} = \tfrac{1}{2}\mathbf{\Xi}_{\text{sym}} : (\mathbf{C}^{-1}\dot{\mathbf{C}} + \dot{\mathbf{C}}\mathbf{C}^{-1}). \tag{82}$$

Proof. *If for the above relations to hold, one must have, in indicial notation,*

$$\dot{Y}_{IJ} = \tfrac{1}{2}(G_{IK}(U^{-1})^n_J \dot{U}^K_n + G_{JK}(U^{-1})^n_I \dot{U}^K_n). \tag{83}$$

Clearly, $\mathbf{Y}$ can be understood as a function of $\mathbf{U}$. Accordingly we have

$$\dot{Y}_{IJ} = \frac{\partial Y_{IJ}}{\partial U^L_k}\dot{U}^L_k, \tag{84}$$

with the help of which, Eq. (83) can be converted in a partial differential equation resulting in

$$\frac{\partial Y_{IJ}}{\partial U^L_k} = \tfrac{1}{2}(G_{IL}(U^{-1})^k_J + G_{JL}(U^{-1})^k_I). \tag{85}$$

G_{IL} are the covariant components of the metric at the reference configuration.

The integrability condition of the last equation reads

$$\frac{\partial Y_{IJ}}{\partial U_r^S \partial U_k^L} = \frac{\partial Y_{IJ}}{\partial U_k^L \partial U_r^S}. \tag{86}$$

With

$$\frac{\partial (U^{-1})^i_J}{\partial U_r^S} = -(U^{-1})^r_J (U^{-1})^i_S \tag{87}$$

one arrives at the integrability condition

$$[G_{IL}(U^{-1})^r_J + G_{JL}(U^{-1})^r_I](U^{-1})^k_S - [G_{IS}(U^{-1})^k_J + G_{JS}(U^{-1})^k_I](U^{-1})^r_L = 0. \tag{88}$$

Since the components of **U** as far as the symmetry is not violated can be chosen arbitrary, the last equation cannot hold, as a simple choice of indices immediately shows.

Accordingly, Σ or $\boldsymbol{\Xi}_{\text{sym}}$ do not have any dual variable, in the sense stated above. Moreover, the result infers immediately that the statement holds true for the Kirchhoff or Cauchy stress tensor themselves. □

4. Discussion

The negative result of the last subsection motivates the search for a non-classical approach to strain measures which may reveal the dual variables of the Cauchy stress tensor and its two material counterparts.

Given a material time rate of a classical strain tensor, say $\dot{\mathbf{C}}$, we observe first that one can construct the corresponding tensor by an integration over time according to

$$\mathbf{C} = \mathbf{1} + \int_{t_0}^{t} \dot{\mathbf{C}}\, \mathrm{d}t^*. \tag{89}$$

We have assumed that **1** is the value of **C** at t_0. While such a relation appears trivial as far as the rate of a well defined strain tensor is concerned, it can be extended to construct non-trivial strain tensors when only the rates are given.

Starting from Eq. (19) we define first a new strain tensor in the following way

$$\boldsymbol{\rho} = \tfrac{1}{2} \int_{t_0}^{t} (\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1})\mathrm{d}t^*. \tag{90}$$

Accordingly, the tensor $\boldsymbol{\rho}$ is the accumulation of the corresponding rate over a specific period of time. From the definition follows

$$\dot{\boldsymbol{\rho}} = \tfrac{1}{2}(\mathbf{U}^{-1}\dot{\mathbf{U}} + \dot{\mathbf{U}}\mathbf{U}^{-1}), \tag{91}$$

which, at a first glance, would provide us with the conjugate strain we are looking for. We assume, without loss of generality, that $\boldsymbol{\rho} = \mathbf{0}$ at $t = t_0$.

Now, from the negative result established in the last subsection, we know that this new strain tensor cannot be formulated as a function of **U** or **C**. However, the constructed strain tensor is not a strain measure in the classical sense as it is path-dependent and, accordingly, of less interest in conjunction with an internal potential.

The same technique can be used to construct a pseudo-dual variable for Ξ. One defines now the strain measure

$$\boldsymbol{\eta} = \frac{1}{2}\int_{t_0}^{t} (\mathbf{C}^{-1}\dot{\mathbf{C}})\,\mathrm{d}t^*, \tag{92}$$

which directly gives

$$\dot{\boldsymbol{\eta}} = \tfrac{1}{2}\mathbf{C}^{-1}\dot{\mathbf{C}}. \tag{93}$$

Here also the path-dependency of this quantity prevents its consideration as a useful strain measure.

5. Closure

In this paper fundamental issues concerning the logarithmic strain tensor, the Cauchy stress tensor and its material counter part, the rotated stress tensor and the Eshelby-like one, have been discussed.

The conjugate stress of the logarithmic strain was derived and new expressions relating it to the rotated stress and the Eshelby-like one are given. It was proven, that the commutator of the logarithmic strain and its conjugate variable determines the skew-symmetric part of the Eshelby-like tensor. In the isotropic case, the conjugate variables of the aforementioned stress tensors are derived in the case of hyperelasticity as well as in the case of multiplicative finite strain viscoplasticity.

Contrasting this, it is rigorously proven that in the general anisotropic case no conjugate strain tensors exist within the concept of classically defined strain measures.

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